Fourier Analysis of Musical Intervals

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Michael C Lopresto
University of Michigan

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Michael C. LoPresto, Henry Ford Community College, Dearborn, MI

Use of a microphone attached to a computer to capture musical sounds and software to display their waveforms and harmonic spectra has become somewhat commonplace.[1] A recent article in *The Physics Teacher* aptly demonstrated the use of MacScope[2] in just such a manner as a way to teach Fourier analysis.[3] A logical continuation of this project is to use MacScope not just to analyze the Fourier composition of musical tones but also musical intervals.

This exercise does not involve complex mathematics, so it can be useful as either a demonstration or experiment showing Fourier series as the basis of musical intervals in a more qualitative musical acoustics or science of sound course, the type of course for which it was developed. It has, however, also been used in a more mathematically advanced introductory physics course as a demonstration and example of a practical application of Fourier analysis.

Musical intervals have been recognized since the time of Pythagoras as combinations of musical pitches. Different intervals are identified by the ratios of the frequencies of the two pitches involved. The frequency of a sound wave is the main cause of the ear's perception of musical pitch.[4] When two pitches are sounded simultaneously, the human eardrum actually vibrates at both frequencies, and the brain perceives the interval as a single musical pitch with a frequency that is the fundamental of a harmonic series to which both pitches belong. This pitch, known as a "missing fundamental" because it is not really there, is heard with a quality (or, as musicians call it, "timbre") that results from the harmonic spectrum and resulting complex waveform of the combination of the two tones. In this process, known as "fundamental tracking," it is as if the ear is actually performing a Fourier-transform.[5]

Table I shows the pitches, measured with MacScope (see Fig. 1), of the middle octave of a grand piano and the frequency ratios they make with the first note of the scale, middle C. The whole-number ratios are those that most closely correspond to each interval. The name of each interval is based on its position in the scale, not the numbers in the ratio. Note that most of the whole-number ratios, although not all,[6] are those of the "Just" musical scale, on which most sets of laboratory tuning forks are based. There are many different possible scales or "intonations" and different frequency-ratios that can be used for each interval,[7] but it has been noted, again, since the time of Pythagoras, that the ear seems to respond most favorably to intervals with smaller whole-number ratios.

Musical pitches consisting of only a fundamental, like those produced by a vibrating tuning fork, combine in a straightforward manner. The spectrum of the combination tone will have two harmonics and the resulting waveform will be more complex, repeating itself with a frequency equal to the above mentioned "missing fundamental." For example, from a standard set of laboratory tuning forks, when C-256 and G-384 are sounded together, an interval of 384/256=3/2, a perfect fifth results. Individually,

Table I. Frequencies of Musical Notes Measured with MacScope.

Musical Note	Frequency (measured) Hz	Ratio of frequency of note to C	Musical Interval	Closest whole-number ratio for interval	f_1 (exp.) Hz	f_1 (meas.) Hz
C	261	1	unison	1		Fig. 1
C# (Db)	278	1.07	(minor) m2nd	16/15	17.4	
D	295	1.13	(major) M2nd	9/8	32.8	32.8
D# (Eb)	311	1.19	m3rd	6/5	51.8	51.6 (Fig. 3)
E	330	1.26	M3rd	5/4	66	66.1
F	348	1.33	(perfect) P4th	4/3	87	87.1
F# (Gb)	369	1.41	(tritone)[6]	7/5	52.7	52.6 (Fig. 6)
G	392	1.5	P5th	3/2	131	131 (Fig. 2)
G# (Ab)	414	1.59	m6th	8/5	51.8	51.8
A	439	1.68	M6th	5/3	87.8	87.8 (Fig. 4)
A# (Bb)	467	1.79	m7th	16/9	29.2	29.4 (Fig. 5)
B	493	1.89	M7th	15/8	32.9	32.8
C	523	2.0	P8th (octave)	2/1	262	262

Table I: The frequencies of each musical note in the middle octave of a grand piano measured with MacScope, the ratios of each musical interval in the octave calculated with the measured frequencies, and the closest whole-number ratio for each interval. The last two columns are the expected frequencies of the "missing fundamental" of the harmonic series for each interval compared with those measured with MacScope. The "missing fundamental" for the interval of a minor second was not recorded because not enough of the long waveform was captured with MacScope to measure it.

each tuning fork produces a "dull" simple sounding sine-wave of their given frequency, but their combination has a more interesting quality and a frequency of 384/3=256/2=128. This can be tested with the aid of computer hardware and software[1,2] for any pair of tuning forks in a standard laboratory set.[8]

For voices, a piano, or other musical instruments, each individual tone has multiple harmonics and therefore its own complex waveform. This makes combinations of these tones even more complex. Figure 2 is the display from MacScope of the waveform and harmonic spectrum of an interval of a fifth, the C and G keys of the piano played simultaneously. Note that the first harmonic, the above-defined "missing fundamental" is barely, if at all, there (see next paragraph), and the first significant harmonics pres-

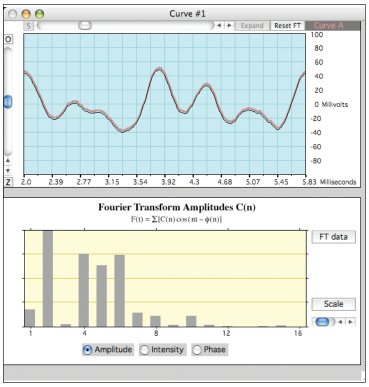

Fig. 1. The display from MacScope showing one cycle of the waveform of the note middle C from a grand piano and its Fourier spectrum. MacScope calculates the fundamental frequency of the cycle selected; these were recorded for all pitches in Table I.

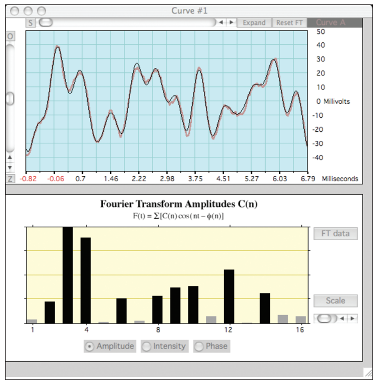

Fig. 2. The waveform of the interval of a fifth, C-G, displayed on MacScope and its Fourier spectrum. Note the dominance of the 2nd and 3rd harmonics and their multiples. The closest whole-number ratio for the fifth is 3/2. The highlighted harmonics are those present in the dark outline of the waveform superimposed on the thicker-red waveform captured by MacScope.

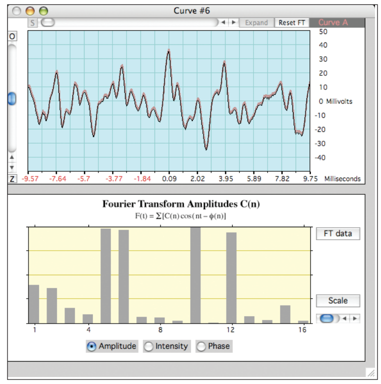

Fig. 3. The waveform and spectrum of the interval of a minor third, C-Eb (6/5), showing the dominance of the 5th and 6th and their multiples the 10th and 12th harmonics.

ent are the second and the third, the numbers from the interval's whole-number ratio, 3/2. Also note the higher harmonics present (4, 6, 8, 9, 10, 12, and 14), all multiples of 2 or 3 or both, and several pairs (6/4, 9/6, and 12/8), all in the ratio 3/2.

The apparent presence of harmonics (with small amplitudes) that are not multiples of the numbers in the closest whole-number ratio is likely due to the fact that it is difficult to select exactly one cycle of the waveform with MacScope on which to perform the Fourier transform. The program treats the portion of the waveform selected as exactly one complete cycle, so inclusion or exclusion of a small amount of the curve beyond or short of an exact cycle will cause the inclusion of "spurious" harmonics in the spectrum.[3] These, however, are not difficult to identify as they will have small amplitudes, and a waveform very close to the one displayed can be constructed with MacScope, as shown in Fig. 2. without them. This difficulty in selecting exactly one cycle of the waveform also may result in small errors in measurement of the frequencies of the "missing fundamentals" of the intervals. However, selecting slightly different amounts of the display as one waveform did not result in significantly different harmonic spectra. The dominant harmonics of the spectra were always the same. The last two columns of Table I compare the expected values of the "missing fundamental," which is determined by dividing the higher frequency in the interval by the higher number in its closest whole-number ratio to that measured from the waveform of the interval from the MacScope display.

The waveforms of all of the intervals in Table I were captured and displayed with Macscope. Their harmonic spectra all showed dominance of the harmonic numbers equal to those that appear in their closest whole-number ratio and their multiples. The interval of a fifth C-G, along with the fourth C-F and the octave C-C′, has relatively simple waveforms compared to others. These intervals are often called the perfect consonances. The waveforms of the major and minor thirds and sixths are more complex and are often referred to as imperfect consonances. Figures 3 and 4 show the waveforms and harmonic spectra of the minor third C-Eb and major sixth C-A, respectively. The seconds and sevenths and the tritone[6] have the most complex waveforms and are called dissonant intervals.[9] Figures 5 and 6 feature dissonant intervals.

If a "field trip" to the nearest grand piano is not practical (ours is in the choir room in the Fine Arts

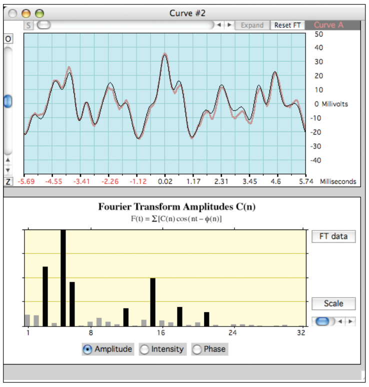

Fig. 4. The waveform and spectrum of a major-sixth, C-A (5/3), showing the dominance of the 3rd and 5th harmonics and many of the higher multiples of the pair. As in Fig. 2 the highlighted harmonics are those present in the dark outline of the waveform.

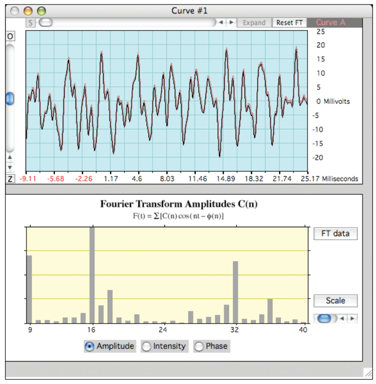

Fig. 5. The waveform and spectrum of the dissonant interval, a minor-seventh, C-Bb (16/9), showing the dominance of the 9th and 16th harmonics and many of the higher multiples of the pair.

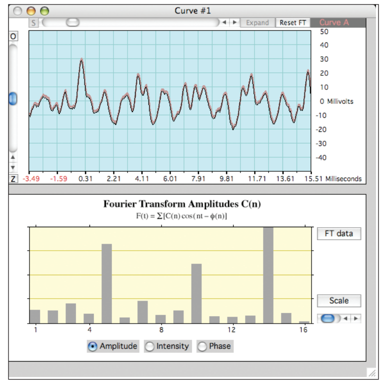

Fig. 6 The waveform and spectrum of the "tritone" interval, C-Gb (F#). The closest whole-number ratio for the tritone is 7/5. This is not what the tritione is considered to be in Just intonation (as are all the other ratios given in Table I),[8] but notice the dominance in the spectrum of the 5th and 7th and their multiples the 10th and 14th harmonics.

Building, just a short walk, lap-tops in hand, from the Science Building), a more portable, upright piano (usually on wheels) or even an electronic piano could be used.

Acknowledgements

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References

1. Both Vernier Software, http://www.vernier.com, and Pasco-Data Studio, http://www.pasco.com, feature software and hardware for this purpose.
2. Download MacScope and directions for its use, free, at http://www.Physics2000.com. Despite the name, both Macintosh and PC versions are available.
3. E. Huggins, "Teaching Fourier analysis in introductory physics," *Phys. Teach.* **45**, 26–29 (Jan. 2007).
4. T.D. Rossing, F.R. Moore, and P.A. Wheeler, *The Science of Sound*, 3rd ed. (Addison-Wesley, San Francisco, 2002), p. 95.
5. J.G. Roederer, *The Physics and Psychophysics of Music– An Introduction*, 3rd ed. (Springer-Verlag, NY, 1995), p. 47.
6. The augmented-fourth or diminished-fifth, also known as the "tritone" in Just intonation, is the ratio 45/32. The reason for the use of 7/5, the "septimal" tritone, can be seen in Fig. 6. See W. A. Sethares, *Tuning, Timbre, Spectrum Scale,* 2nd ed. (Springer, London, 2005), p.101.
7. See Partch's 43-tone scale, Ref. 6, p. 62.
8. M.C. LoPresto, "Experimenting with musical intervals," *Phys. Educ.* **38**, 309–315 (July 2003).
9. Ref. 5, p. 166.

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Michael C. LoPresto *is currently serving as physics department chair and planetarium director at HFCC.*
Henry Ford Community College, Dearborn, MI 48128; lopresto@hfcc.edu